

Quantum simulator for the Ising model with electrons floating on a helium film

Sarah Mostame¹ and Ralf Schützhold^{1,2,*}

¹*Institut für Theoretische Physik, Technische Universität Dresden, 01062 Dresden, Germany*

²*Fachbereich Physik, Universität Duisburg-Essen, D-47048 Duisburg, Germany*

We propose a physical setup that can be used to simulate the quantum dynamics of the Ising model in a transverse field with present-day technology. Our scheme consists of electrons floating on superfluid helium which interact via Coulomb forces. In the limit of low temperatures, the system will stay near its ground state where its Hamiltonian is equivalent to the Ising model and thus shows phenomena such as quantum criticality. Furthermore, the proposed design could be generalized in order to study interacting field theories (e.g., $\lambda\phi^4$) and adiabatic quantum computers.

PACS numbers: 03.67.Ac; 75.10.Hk.

Introduction Richard Feynman's observation [1] that classical computers cannot effectively simulate quantum systems bred widespread interest in quantum computation. He thought up the idea of a quantum processor which uses the effects of quantum theory instead of classical physics. As an example, Feynman proposed a *universal quantum simulator* consisting of a lattice of spins with nearest neighbor interactions that are freely specifiable and can efficiently reproduce the dynamics of *any* other many-particle quantum system with a finite-dimensional state space [1]. Although such universal quantum computers of sufficient size (e.g., number of Qubits, i.e., spins) are not available yet, it is possible to design a special quantum system in the laboratory which simulates the quantum dynamics of a particular model of interest. Such a designed quantum system can then be regarded as a special quantum computer (instead of a universal one, which is more challenging) which just performs the desired quantum simulation, see, e.g., [2, 3, 4].

In the following, we present a design for a quantum simulator for the Ising spin chain in a transverse field and demonstrate that it could be feasible with present-day technology, i.e., electrons floating on a thin superfluid Helium film. A similar idea based on trapped ions has been pursued in [2]. Nevertheless, since different experimental realizations possess distinct advantages and drawbacks, it is still worthwhile to study an alternative set-up. For example, the number of coherently controlled ions in a trap is rather limited at present, whereas our proposal can be scaled up to a large number of electrons more easily – which is important for exploring the continuum limit and scaling properties etc.

The model We want to simulate the quantum dynamics of the one-dimensional Ising chain consisting of n spins with nearest-neighbor interaction J plus a transverse field Γ along the x -direction ($\hbar = 1$)

$$H = - \sum_{j=1}^n \{ \Gamma \sigma_j^x + J \sigma_j^z \sigma_{j+1}^z \}, \quad (1)$$

where $\sigma_j = (\sigma_j^x, \sigma_j^y, \sigma_j^z)$ are the spin-1/2 Pauli matrices acting on the j th qubit. This model has been employed in

the study of quantum phase transitions and percolation theory [5], spin glasses [5, 6], as well as quantum annealing [7, 8] etc. Although the Hamiltonian (1) is quite simple and can be diagonalized analytically, the Ising model is considered a paradigmatic example [5] for second-order quantum phase transitions and is rich enough to display most of the basic phenomena near quantum critical points. For $\Gamma \gg J$, the ground state is paramagnetic $|\rightarrow\rightarrow\rightarrow\ldots\rangle$ with all spins polarized along the x axis. In the opposite limit $\Gamma \ll J$, the nature of the ground state(s) changes qualitatively and there are two degenerate ferromagnetic phases with all spins pointing either up or down along the z axis $|\uparrow\uparrow\uparrow\ldots\rangle$ or $|\downarrow\downarrow\downarrow\ldots\rangle$. The two regimes are separated by a quantum phase transition at the critical point $\Gamma_{\text{cr}} = J$, where the excitation gap vanishes (in the thermodynamic limit $n \uparrow \infty$) and the response time diverges. As a result, driving the system through its quantum critical point at a finite sweep rate entails interesting non-equilibrium phenomena such as the creation of topological defects, i.e., kinks [9]. Furthermore, the transverse Ising model can also be used to study the order-disorder transitions at zero temperature driven by quantum fluctuations [5, 7]. Finally, two-dimensional generalizations of the Ising model can be mapped onto certain adiabatic quantum algorithms (see, e.g., [10]). However, due to the evanescent excitation energies, such a phase transition is rather vulnerable to decoherence, which must be taken into account [11].

The analogue In order to reproduce the quantum dynamics of the 1+1 dimensional Ising model (1), we propose trapping a large number of electrons on a low-temperature helium film of thickness h (e.g., $h = 110$ nm) adsorbed on a silicon substrate [12]. Due to the polarizability $\varepsilon \approx 1.06$ of the Helium film, the electrons are bound to its surface (i.e., in z -direction) via their image charges and the large potential barrier (around 1 eV) for penetration into the helium film [13]. Since the binding energy of around 8 K is much larger than the temperature T (below 1 K) and the width of the electron wave packet in z -direction (of order 8 nm) is much smaller than all other relevant length scales, the electron motion is approximately two-dimensional (x, y -plane).

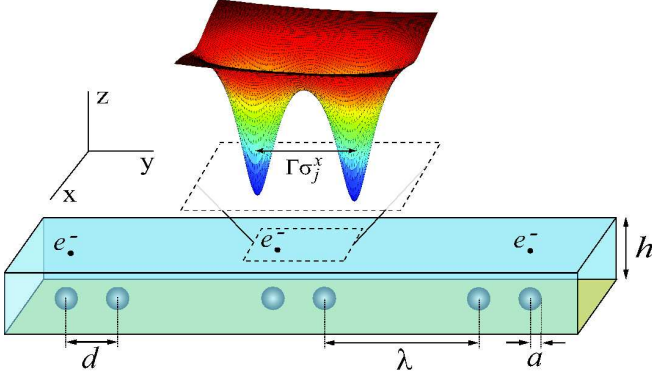


FIG. 1: Sketch of the proposed analogue quantum simulator. Electrons (e^-) are floating on a low-temperature helium film of height h adsorbed on a silicon substrate. A double-well potential for each single electron is created by a pair of golden spheres of radius a and distance d on the bottom of the helium film. The double wells at each site provide two lowest states of the electron and model the spin states $|\uparrow\rangle$ and $|\downarrow\rangle$ at each site j . The tunneling rate between the two wells corresponds to the transverse field term $\Gamma\sigma_j^x$. The electrons are lined up at distances λ and interact via Coulomb forces, which creates the term $J\sigma_j^z\sigma_{j+1}^z$.

In our scheme, each single electron on top of the helium film is trapped by a pair of golden spheres of radius a (e.g., $a = 10$ nm) and distance d (e.g., $d = 60$ nm) attached to the silicon substrate (i.e., on the bottom of the helium film, cf. Fig. 1). Depending on its position x, y , the electron will also induce image charges in the two golden spheres (which act as a pair of quantum dots) and hence experience a double-well potential

$$U_w(x, y) = -\frac{ae^2(x^2 + y^2 + \alpha^2 + \beta^2)/4\pi\epsilon}{(x^2 + y^2 + \alpha^2 + \beta^2)^2 - 4\alpha^2y^2}, \quad (2)$$

with $\alpha = d/2 + a$ and $\beta^2 = h^2 - a^2$. Since this potential is quite deep and symmetric $U(x, y) = U(x, -y)$, cf. Fig. 2, the ground state wave-function $\psi_S(x, y)$ is given by the symmetric superposition of the two Wannier states $\psi_0(x, \pm y)$ while the first excited state $\psi_A(x, y)$ is the anti-symmetric combination

$$\begin{aligned} \psi_S(x, y) &= \frac{\psi_0(x, y) + \psi_0(x, -y)}{\sqrt{2}} \rightarrow \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}, \\ \psi_A(x, y) &= \frac{\psi_0(x, y) - \psi_0(x, -y)}{\sqrt{2}} \rightarrow \frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}}. \end{aligned} \quad (3)$$

For a sufficiently high potential barrier between the two wells, the Wannier state $\psi_0(x, y)$ is strongly concentrated in the left well and models the spin state $|\uparrow\rangle$ and vice versa. The tunneling between the two states is then described by the Pauli operator σ^x with $\sigma^x|\uparrow\rangle = |\downarrow\rangle$ and $\sigma^x|\downarrow\rangle = |\uparrow\rangle$ such that the tunneling rate, given by the difference of the eigenenergies $E_A - E_S$ of ψ_S and ψ_A , corresponds to the transverse field Γ in Eq. (1). In the

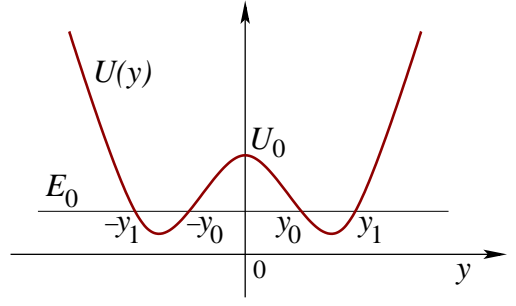


FIG. 2: Sketch of the double-well potential $U(y)$ with four turning points for the energy E_0 .

limit of strong localization (i.e., weak tunneling), the energy splitting $E_A - E_S$ between the two levels can be estimated via the WKB approximation [14]

$$E_A - E_S \approx \frac{\omega}{\pi} \exp \left[- \int_{-y_0}^{y_0} dy |p(y)| \right]. \quad (4)$$

Here ω is the oscillation frequency (within one well) and $\pm y_0$ are the two inner (classical) turning points, cf. Fig. 2. The integrand is given by $p(x, y) = \sqrt{2m_e[E_0 - U(x, y)]}$, where we can set $x = 0$ since the tunneling probability away from the $x = 0$ -axis is strongly suppressed. Finally, the energy E_0 determines the turning points and m_e is the electron mass. For the parameters above, each valley can well be approximated by a harmonic oscillator

$$U_w(x, y \approx \pm y_{\min}) \approx \frac{ae^2}{4\pi\epsilon\beta^4}(x^2 + [y \mp y_{\min}]^2), \quad (5)$$

and thus we obtain $E_0 \approx \sqrt{ae^2/2\pi\epsilon m_e\beta^4} \approx \omega$.

So far, we derived the term $\Gamma\sigma_j^x$ in Eq. (1) via Eqs. (2), (3), (4), and (5). In order to simulate the remaining part, we propose to line up the pairs of quantum dots at equal distances λ (e.g., $\lambda = 600$ nm), where the parameters are supposed to obey the following hierarchy

$$\lambda \gg h > d \gg a. \quad (6)$$

In this limit, the interaction between the electrons will be dominated by the direct Coulomb repulsion between nearest neighbors $U_c(x, y) = \sum_{j=1}^n U_c^{j,j+1}$ with n denoting the number of electrons floating on the helium film. For $\lambda \gg d$, we may Taylor expand the Coulomb interaction into powers of y/λ due to $y \approx \pm d/2$. The zeroth-order term is constant and thus irrelevant while the first-order contributions vanish (up to boundary terms) after the sum over sites j . Thus, the leading term is bilinear in the electron positions

$$U_c(x, y) \approx -\frac{e^2}{2\pi\epsilon_0(\lambda + d + 4a)^3} \sum_{j=1}^n y_j y_{j+1}, \quad (7)$$

and precisely corresponds to the $J\sigma_j^z\sigma_{j+1}^z$ term in Eq. (1)

with the effective coupling

$$J = \frac{e^2(d + 2a)^2}{8\pi\epsilon_0(\lambda + d + 4a)^3}. \quad (8)$$

Experimental parameters For the example values given in the text, we obtain $\Gamma \approx 0.1$ K for the tunneling rate and the same value $J \approx 0.1$ K for the effective coupling, i.e., we are precisely in the quantum critical regime. However, deviations from this critical point should be easy to realize experimentally by varying the height h of the helium film, since the tunneling rate depends strongly (in fact, exponentially) on h , whereas the Coulomb force remains approximately constant. In order to see quantum critical behavior, i.e., to avoid thermal fluctuations, the temperature should ideally be well below this value 0.1 K (or at least not far above it).

Furthermore, the Coulomb repulsion energy between two electrons (zeroth-order term) of about 11 K would tend to destabilize the electron chain. Fortunately, this effect is compensated by the binding energy between the electron and its image on the sphere, which is around 13 K and thus stabilizes the electron chain. The probability for the electron to penetrate the helium film by tunneling to one of the golden spheres is extremely small (of order 10^{-16}) and can be neglected. Finally, the ground-state energy $E_0 \approx 1.4$ K (within the harmonic oscillator approximation) is reasonably well below the barrier height $U_0 \approx 3.1$ K such that the WKB approximation should provide a reasonable estimate. (The tunneling probability of 0.08 is also small enough.) On the other hand, $E_0 \approx 1.4$ K is a measure of the distance between the two lowest-lying states in Eq. (3) and the remaining excited states in the double-well potential. As a result, these additional states do not play a role for temperatures well below one Kelvin and thus the Hamiltonian (1) provides the correct low-temperature description.

Read-out scheme Having successfully simulated the Ising Hamiltonian (1), one is lead to the question of how to actually measure its properties, e.g., how to detect signatures of quantum critical behavior. As one possibility, let us imagine enclosing the Ising chain symmetrically by two electrodes in the form of spheres of radius $R = 100 \mu\text{m}$ and a distance of 1 mm aligned along the chain axis. Applying a voltage of 1 μV , an approximately constant electric field of 4×10^{-4} V/m acts on all the electrons and induces the perturbation Hamiltonian

$$H_{\text{pert}} = \sum_{j=1}^n \gamma \sigma_j^z, \quad (9)$$

corresponding to a longitudinal field (in addition to the transversal one $\Gamma \sigma_j^x$). For $d = 60$ nm, we get $\gamma \approx 0.1 \mu\text{K}$, i.e., a very weak perturbation $\gamma \ll \Gamma$.

Deep in the paramagnetic phase $\Gamma \gg J$, the response of the system to this weak perturbation $\gamma \ll \Gamma$ is rather small $\langle \sigma_j^z \rangle \approx \gamma/\Gamma$. Approaching the phase transition,

however, the static susceptibility $\chi_\gamma = \lim_{\gamma \rightarrow 0} \langle \sigma_j^z \rangle / \gamma$ grows and finally diverges at the critical point. In the broken symmetry phase, the perturbation (9) lifts the degeneracy $\sigma_j^z \rightarrow -\sigma_j^z$ and hence the response is non-analytic, i.e., independent of the smallness of γ : e.g., for $J \gg \Gamma$, we have $\langle \sigma_j^z \rangle = \text{sign}(\gamma) = \pm 1$. This signal $\langle \sigma_j^z \rangle$ indicating the phase transition can be picked up by the two electrodes for which the Ising chain acts like a dielectric medium and induces a voltage shift of order nano-Volt per electron (for $J \gg \Gamma$), which should be measurable for a sufficiently large number of sites. In addition to the static case, one could also study the time-resolved response $\langle \sigma_j^z(t) \rangle$ to a varying voltage $\gamma(t')$, which is determined by the dynamical correlator $\langle \sigma_i^z(t') \sigma_j^z(t) \rangle$ in lowest-order response theory.

Even in the absence of an externally imposed voltage, the chain induces spontaneous voltage fluctuations in the electrodes, which are strongest (of order nano-Volt per electron) deep in the ferromagnetic phase. The variance of these fluctuations yields the correlator sum $\sum_{ij} \langle \sigma_i^z \sigma_j^z \rangle$ which is an order parameter for the phase transition and allows us to detect topological defects (i.e., kinks) which might have been produced during the sweep to the ferromagnetic phase: In the presence of a kink, the ground-state signal $\sum_{ij} \langle \sigma_i^z \sigma_j^z \rangle = n^2$ is drastically reduced (in average to $\sum_{ij} \langle \sigma_i^z \sigma_j^z \rangle = n^2/3$) depending on the kink position. If the kink is precisely in the middle of the Ising chain, we get a vanishing signal $\sum_{ij} \langle \sigma_i^z \sigma_j^z \rangle = 0$, whereas a kink near the boundaries does not diminish the signal strongly.

Disorder and decoherence In a real experimental setup, the Hamiltonian will not be exactly equivalent to (1) due to imperfections such as electric stray fields, variations in the film thickness h and further geometric parameters a , d , and λ etc. Therefore, the original expression (1) will typically be altered to

$$H = - \sum_{j=1}^n \{ \Gamma_j \sigma_j^x + J_j \sigma_j^z \sigma_{j+1}^z + \gamma_j \sigma_j^z \}, \quad (10)$$

where $\Gamma_j = \bar{\Gamma} + \delta\Gamma_j$ and $J_j = \bar{J} + \delta J_j$. Assuming that the disorder parameters $\delta\Gamma_j$, δJ_j , and γ_j are much smaller than the excitation gap $\Delta = 2|J - \Gamma|$ of the undisturbed system (in the continuum limit), the impact of these imperfections will be suppressed. Near the critical point $J \approx \Gamma$, however, this argument fails. Still, for a finite number n of electrons, one retains a minimum gap (within the symmetric or anti-symmetric subspace, respectively) of order J/n . Exploiting this gap might be suitable for a reasonably small systems, but for $n \geq 100$ electrons, the required accuracy on the sub-percent level is probably hard to achieve experimentally. E.g., decreasing the diameter of the golden spheres by ten percent with the other values remaining the same as before, the tunneling rate increases by fifty percent.

For a sufficiently large number of electrons, the disor-

der induced by imperfections will become relevant near the critical point (in one spatial dimension) in view of the critical exponent $\nu = 1$ of the Ising model, see, e.g., [5]. (I.e., the renormalization flow is directed away from the homogeneous situation.) In this case, one would expect effects such as local paramagnetic regions inside the global ferromagnetic phase and percolation transitions etc. Therefore, turning this drawback into an advantage, one might generate these imperfections on purpose in order to study the impact of disorder onto the phase transition. In contrast to the original Hamiltonian (1), the above form (10) is no longer analytically solvable and hence much less is known about its properties. Finally, in a real set-up, the system will also experience decoherence due to the inevitable coupling to the environment [11]. These effects could be incorporated by operator-valued variations $\delta\Gamma_j$, δJ_j , and γ_j associated to the degrees of freedom of the environment – where the same arguments apply as before.

Summary We have proposed a design for the simulation of the quantum Ising model with a system of electrons floating on a liquid helium film adsorbed on a silicon substrate. Since the energy level splitting (tunneling rate Γ) depends exponentially on the thickness of the helium film h , we may tune the system through the quantum phase transition by changing h – which might even be feasible in a time-dependent manner, cf. [9]. The created topological defects (kinks) could be detected via a strong reduction of the spontaneous voltage fluctuations in comparison with the homogeneous ferromagnetic phase.

Furthermore, a suitable generalization to two spatial dimensions might be relevant for adiabatic quantum algorithms, see, e.g., [10]. Note that the realization of a sequential quantum computer based on a set of electrons floating on a helium film has been proposed in [15]. In contrast, our proposal is not suited for universal computations, but (as one would expect) should be easier to realize experimentally.

Exploring a different limit, where many eigenstates of the double-well potential contribute, the proposed set-

up could simulate the lattice version of interacting field theories such as the $\lambda\phi^4$ -model in 1+1 dimensions.

S. M. acknowledges fruitful discussions with R. Farhadifar and R. S. is indebted to G. Volovik and P. Leiderer for valuable conversations. This work was supported by the Emmy-Noether Programme of the German Research Foundation (DFG) under grant SCHU 1557/1-2,3 and by DFG grant SCHU 1557/2-1.

* email: `ralf.schuetzhold@uni-due.de`

-
- [1] R. P. Feynman Int. J. Theor. Phys. **21**, 467 (1986); R. P. Feynman Found. Phys. **16**, 507 (1982).
 - [2] D. Porras and J. I. Cirac, Phys. Rev. Lett. **92**, 207901 (2004).
 - [3] R. Schützhold and S. Mostame, JETP Lett. **82**, 248 (2005).
 - [4] T. Byrnes *et al.*, Phys. Rev. Lett. **99**, 016405 (2007).
 - [5] S. Sachdev, *Quantum Phase transitions*, (Cambridge University Press, Cambridge, UK, 1999).
 - [6] K. H. Fischer, and J. A. Hertz *Spin glasses*, (Cambridge University Press, Cambridge, UK, 1993).
 - [7] A. Das, and B. K. Chakrabarti, *Quantum Annealing and Related Optimisation Methods*, (LNP **679**, Springer-Verlag, Heidelberg 2005).
 - [8] G. E. Santoro *et al.*, Science **295**, 2427 (2002); T. Kadowaki, and H. Nishimori, Phys. Rev. E **58**, 5355 (1998).
 - [9] J. Dziarmaga, Phys. Rev. Lett. **95**, 245701 (2005).
 - [10] IEEE Spectrum online, Tech Talk, February 13th (2007).
 - [11] S. Mostame, G. Schaller, and R. Schützhold, Phys. Rev. A **76**, R030304 (2007).
 - [12] J. Angrik *et al.*, Journal of Low Temperature Physics, **137**, 335 (2004).
 - [13] E. Y. Andrei, Ed., *Two Dimensional Electron Systems on Helium and Other Cryogenic Substrates*, (Academic Press, New York, 1991).
 - [14] M. Razavy, *Quantum theory of tunneling*, (World Scientific, 2003).
 - [15] P.M. Platzman and M.I. Dykman, Science **284**, 1967 (1999).